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# A VERSION OF A REFINED NON-LINEAR THEORY OF THIN ELASTIC SANDWICH SHELLS OF ITERATION TYPE*

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A version is proposed for the refined non-linear theory of thin elastic sandwich shells of a quadratic approximation based on reliance on the traditional Kirchhoff-Love hypotheses on the outer layers and a refined model on the filler. An iteration procedure is used to construct the latter, within whose framework expressions are derived for the components of the displacement vector in the first stage, under the assumption that the filler is transversely soft, by successive integration of the relationships of the three-dimensional theory of thermoelasticity with respect to the transferse coordinate. These expressions are then used in the second stage to calculate the components of the strain and stress tensor yielding to refinement. The advisability of using the set of relationships obtained to investigate the stability of sandwich shells in the refined formulation in order to clarify mixed buckling modes of the outer layers and the shell as a whole, is noted.

The refined classification of buckling modes of sandwich plates and shells /1/ includes, in addition to the cophasal (skew-symmetric) and antiphasal (symmetric) modes described and studied in the literature /2, 3/, a mixed mode of outer layer buckling. The equations proposed in /1/ for investigating mixed buckling modes and used (in /4/, etc.) to solve specific problems and based on the static-kinematic model of a "broken" line /3/, are ultimately simplified. They required refinement in order to formulate stability problems, firstly for those sandwich structures for which the thicknesses of the outer layers $(2h_{(k)},\ k=1,\ 2)$ and the filler $(2h)$ satisfy the conditions $h_{(k)}/h\sim\varepsilon$, where $\varepsilon$ is a certain small quantity that can be neglected compared with unity (sandwich structure with quite thin outer layers). In this connection, a modification is proposed of the relationships derived in /5/ based on using the iteration procedure to refine the state of stress and strain in the filler. Temperature effects are also taken into account within the framework of the relationships of uncoupled thermoelasticity.

1. As in /5/, we refer the spaces occupied by the filler and the outer layers to the parametrizations

$$\rho(\alpha^i, z)=\mathbf{r}(\alpha^i)+z\mathbf{m},\quad \rho_{(k)}(\alpha^i, z_{(k)})=\mathbf{r}_{(k)}+z_{(k)}\mathbf{m}$$
$$\mathbf{r}_{(k)}=\mathbf{r}-\delta_{(k)}(h+h_{(k)})\,\mathbf{m},\quad -h\leqslant z\leqslant h,\quad -h_{(k)}\leqslant z_{(k)}\leqslant h_{(k)}$$

where $\mathbf{r}$ is the radius-vector of a point $M$ on the middle surface of the filler $\sigma$ referred to a curvilinear system of coordinates $\alpha^i$ and characterized by the metric tensors $a_{ij}=\mathbf{r}_i\mathbf{r}_j$, $b_{ij}=-\mathbf{r}_i\mathbf{m}_j$; $\mathbf{m}$, $\mathbf{m}_j=\partial\mathbf{m}/\partial\alpha^j$ is the unit normal vector to $\sigma$ and its derivative with respect to $\alpha^j$; $2h$, $2h_{(k)}$ are the filler and carrying layer thicknesses $(k=2$ for the upper layer and $k=1$ for the lower layer), $\delta_{(k)}$ is a symbol that takes the integer values $\delta_{(1)}=1$ and $\delta_{(2)}=-1$, and $\mathbf{r}_{(k)}$ are radius-vectors of points on the middle surfaces of the outer layers $\sigma_{(k)}$. We consider the filler and outer layers to be thin by assuming the following approximate equalities to be satisfied

$$\delta_i^j-h_{(k)}b_i^j\approx\delta_i^j,\quad \delta_i^j-hb_i^j\approx\delta_i^j$$

($\delta_i^j$ are Kronecker deltas), consequently, we later identify the basis vectors $\mathbf{r}_i=\partial\mathbf{r}/\partial\alpha^i$ on $\sigma$ and $\rho_i=\partial\rho/\partial\alpha^i$, $\rho_i^{(k)}=\partial\rho_{(k)}/\partial\alpha^i$.

We take the Kirchhoff-Love hypotheses transitional in the theory of sandwich shells /3/ to describe the mechanics of outer-layer deformation, and within their framework express the

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displacement vector $\mathbf{U}^{(k)}$ of the $k$-th outer layer and the covariant components of its tangential strain tensor during average shell bending by the equalities /6/ ($\nabla_i$ are covariant differentiation operators in the metric $a_{ik}$)

$$\mathbf{U}^{(k)} = u_i^{(k)}\mathbf{r}^i + w^{(k)}\mathbf{m} - z_{(k)}\omega_i^{(k)}\mathbf{r}^i \tag{1.1}$$

$$\begin{gathered}\varepsilon_{ij}^{z(k)} = \varepsilon_{ij}^{(k)} + z_{(k)}\varkappa_{ij}^{(k)}, \quad 2\varepsilon_{ij}^{(k)} = e_{ij}^{(k)} + e_{ji}^{(k)} + \omega_i^{(k)}\omega_j^{(k)} \\ 2\varkappa_{ij}^{(k)} = -\nabla_i\omega_j^{(k)} - \nabla_j\omega_i^{(k)}, \quad e_{ij}^{(k)} = \nabla_i u_j^{(k)} - b_{ij}w^{(k)} \\ \omega_i^{(k)} = \nabla_i w^{(k)} + b_i^{\,j}u_j^{(k)}\end{gathered} \tag{1.2}$$

In order to establish a law for the variation of the filler displacement vector component $\mathbf{U} = U_i\mathbf{r}^i + U_3\mathbf{m}$ along the $z$ coordinate, we make the assumption

$$\sigma^{11} = \sigma^{12} = \sigma^{22} = 0 \tag{1.3}$$

for its shear stress tensor components $\sigma^{ij}$ by considering the filler transversely soft at this stage.

On using these equalities, the successive integration of the equilibrium equations written for the filler to the accuracy of $\delta_i^{\,j} - zb_i^{\,j} \approx \delta_i^{\,j}$ and neglecting bulk forces therein, we obtain the formulas /5/

$$\sigma^{33} = q_{(1)}^{\;3} - (z+h)\,\nabla_i q^i, \; q_{(2)}^{\;3} = q_{(1)}^{\;3} - 2h\nabla_i q^i \tag{1.4}$$

where $q^i = \sigma^{i3}(\alpha^i)$ are transverse shear stresses independent of $z$, and $q_{(k)}^3 = \sigma^{33}(\alpha^i,\; z = -\delta_{(k)}h)$ are values of the transverse compressive stresses at points of the surface $z = -\delta_{(k)}h$.

Within the framework of the same degree of accuracy given by the approximate equalities (1.3) and $\delta_i^{\,j} - zb_i^{\,j} \approx \delta_i^{\,j}$, the equation of state for $\sigma^{33}$ corresponding to relationships of the uncoupled problem of thermoelasticity will, when the first formula in (1.4) is taken into account, have the form

$$\sigma^{33} = E_3\,(\partial U_3/\partial z - \alpha_3 T) = q_{(1)}^3 - (z+h)\,\nabla_i q^i \tag{1.5}$$

not only for small but also for average shell bending. Here $E_3$, $\alpha_3$ are the elastic modulus and coefficient of thermal expansion of the filler in the $z$ direction, and $T$ is the temperature increment.

Integration of this equation and subsequent satisfaction of the kinematic conditions for connecting the outer layers to the filler according to deflections

$$U_3\,|_{z=-h} = w^{(1)}, \quad U_3\,|_{z=h} = w^{(2)}$$

results in the formulas

$$q_{(k)}^3 = \varphi_3\,(w^{(2)} - w^{(1)}) + \delta_{(k)}h\nabla_i q^i - \varphi_3\beta_3 \tag{1.6}$$

$$\varepsilon_{33} = \frac{w^{(2)} - w^{(1)}}{2h} - \frac{z}{E_3}\nabla_i q^i - \frac{\beta_3}{2h} = \alpha_3 T \tag{1.7}$$

$$U_3 = \frac{w^{(1)} + w^{(2)}}{2} + \frac{z}{2h}\,(w^{(2)} - w^{(1)}) - \frac{z^2 - h^2}{2E_3}\nabla_i q^i - \frac{z+h}{2h}\beta_3 + \lambda_3, \tag{1.8}$$

$$\beta_3 = \int_{-h}^{h}\alpha_3 T\,dz, \quad \varphi_3 = \frac{E_3}{2h}, \quad \lambda_3 = \int_{-h}^{z}\alpha_3 T\,dz$$

where $\varepsilon_{33}$ is the deformation of transverse compression of the filler.

To establish a law of the change of the tangential components of the displacement in the filler with respect to $z$, we turn to the equations of state for $\sigma^{i3}$. In the case of average bending for a filler with elastic properties symmetric with respect to the surfaces $z = \text{const}$, these equations can be represented in the following approximate form:

$$\begin{gathered}\sigma^{i3} = q^i = 2A^{ik}\varepsilon_{k3} = A^{ik}\left[(\delta_k^{\,s} - zb_k^{\,s})\,\partial U_s/\partial z + \partial U_3/\partial\alpha^k + b_k^{\,s}U_s\right] \approx \\ A^{ik}\,(\partial U_k/\partial z + \nabla_k U_3)\end{gathered} \tag{1.9}$$

where $A^{ik}$ is the bivalent tensor of the filler shear elastic constants. Starting from (1.8) and (1.9), we arrive at a different equation to determine $U_k$

$$\begin{gathered}\frac{\partial U_k}{\partial z} = d_{ki}q^i - \frac{\omega_k^{(1)} + \omega_k^{(2)}}{2} - \frac{z}{2h}\,(\omega_k^{(2)} - \omega_k^{(1)}) + \\ \frac{z^2 - h^2}{2E_3}\nabla_k\nabla_i q^i + \frac{z+h}{2h}\nabla_k\beta_3 - \nabla_k\lambda_3\end{gathered} \tag{1.10}$$

in which the quantities $\nabla_i w^{(k)}$ are replaced by the quantities $\omega_i^{(k)}$ to the assumed degree of accuracy apparent when substituting relationships (1.8) into (1.9).

Furthermore, integrating (1.10) with respect to $z$ we obtain

$$U_k = u_k + d_{ki} z q^i - z\frac{\omega_k^{(1)} + \omega_k^{(2)}}{2} - \frac{z^2}{4h}(\omega_k^{(2)} - \omega_k^{(1)}) + \frac{1}{2E_3}\left(\frac{z^3}{3} - h^2 z\right)\nabla_k \nabla_i q^i + \frac{1}{2h}\left(\frac{z^2}{2} + hz\right)\nabla_k \beta_3 - \nabla_k \Lambda_3 \tag{1.11}$$

$$\Lambda_3 = \int_0^z \lambda_3\, dz, \quad u_k = U_k|_{z=0}$$

Satisfying relationships (1.1) and (1.11) for the connection conditions of the outer layers to the filler in tangential displacements

$$u_i^{(1)} - h_{(1)}\omega_i^{(1)} = U_i|_{z=-h}, \quad u_i^{(2)} + h_{(2)}\omega_i^{(2)} = U_i|_{z=h}$$

we arrive at a system of two algebraic equations from which the relations

$$u_i = \frac{u_i^{(1)} + u_i^{(2)}}{2} + \left(\frac{h_{(2)}}{2} + \frac{h}{4}\right)\omega_i^{(2)} - \left(\frac{h_{(1)}}{2} + \frac{h}{4}\right)\omega_i^{(1)} + \frac{h}{4}\nabla_i \beta_3 + \frac{1}{2}\nabla_i(\Lambda_3^+ + \Lambda_3^-), \quad \Lambda_3^{\pm} = \Lambda_3|_{z=\pm h}$$

$$^2/_3 h^3 E_3^{-1} \nabla_n \nabla_i q^i = u_k^{(1)} - u_k^{(2)} + c_{ki} q^i - (h_{(1)} + h)\omega_k^{(1)} - (h_{(2)} + h)\omega_k^{(2)} + h\nabla_k \beta_3 - \Lambda_k, \quad c_{ki} = 2hd_{ki}$$

$$\Lambda_k = \int_{-h}^{h} \nabla_k\left(\int_{-h}^{z} \alpha_3 T\, dz\right) dz$$

are established. Substituting these relations into (1.11), we obtain

$$U_i = f_{(1)} u_i^{(1)} + f_{(2)} u_i^{(2)} + \chi_{(1)}\omega_i^{(1)} + \chi_{(2)}\omega_i^{(2)} + \chi\, d_{is} q^s + U_i^{(T)} \tag{1.12}$$

$$f_{(k)} = \frac{1}{2} + \delta_{(k)}\frac{z^3}{4h^3} - \delta_{(k)}\frac{3z}{4h}, \quad \chi = z + \frac{3}{2h^2}\left(\frac{z^3}{3} - h^2 z\right)$$

$$\chi_{(k)} = -\delta_{(k)}\left(\frac{h_{(k)}}{2} + \frac{h}{4}\right) - \frac{z}{2} + \delta_{(k)}\frac{z^2}{4h} - \frac{3(h_{(k)} + h)}{4h^3}\left(\frac{z^3}{3} - h^2 z\right)$$

$$U_k^{(t)} = \frac{h}{4}\nabla_k \beta_3 + \frac{1}{2h}\left(\frac{z^2}{2} + hz\right)\nabla_k \beta_3 - \nabla_k \Lambda_3 + \frac{3}{4h^3}\left(\frac{z^3}{3} - h^2 z\right)(h\nabla_k \beta_3 - \Lambda_k) + \frac{1}{2}\nabla_k(\Lambda_3^+ + \Lambda_3^-)$$

As follows from (1.8) and (1.5), the displacement field in the filler is determined in terms of eight two-dimensional functions $u_i^{(k)}, w^{(k)}, q^i$ that appear in a natural way during integration of the three-dimensional relationships of thermoelasticity with respect to $z$ within the framework of the assumptions (1.8) made and the further satisfaction of the kinematic connection conditions for the outer layers and the filler. Subsequent utilization of relationships (1.8) and (1.12), that hold for average shell bending, enables us to evaluate the tangential components of the strain tensor in the filler, which turn out to be the following in a linear approximation with accuracy $\delta_i^k - zb_i^k \approx \delta_i^k$

$$2\varepsilon_{ik} = \mathbf{r}_k \partial_i \mathbf{U} + \mathbf{r}_i \partial_k \mathbf{U} = E_{ik} + E_{ki} \tag{1.13}$$

$$E_{ik} = \nabla_i U_k - b_{ik} U_3 = f_{(1)}\nabla_i u_k^{(1)} + f_{(2)}\nabla_i u_k^{(2)} + \chi_{(1)}\nabla_i \omega_k^{(1)} + \chi_{(2)}\nabla_i \omega_k^{(2)} + \chi \nabla_i q^s d_{ks} + \nabla_i U_k^{(T)} - b_{ik}\left[\frac{w^{(1)} + w^{(2)}}{2} + \frac{z}{2h}(w^{(2)} - w^{(1)}) - \frac{z^2 - h^2}{2E_3}\nabla_s q^s + (z + h)\beta_3/2h^{-1} + \lambda_3\right]$$

and then, dropping the initial simplifying assumptions (1.3) at this stage, calculate the shear stress tensor components from the formulas

$$\sigma^{ik} = A^{ikjs}\varepsilon_{js} + A^{ik33}\varepsilon_{33} - \beta^{ik}T \tag{1.14}$$

$$\sigma^{33} = A^{3333}\varepsilon_{33} + A^{ik33}\varepsilon_{ik} - \beta^{33}T$$

and refine the transverse compressive stress in the filler as compared with (1.5).

The procedure elucidated for constructing a refined elastic deformation model of a sandwhich shell is an iteration type, and the limits of its applicability are determined by the limit of applicability of the assumption $\sigma^{i3} = q^i(\alpha^k)$, that remains valid and occurs exactly only under strict compliance with the equalities (1.3). It differs from the iteration procedure for constructing refined models in the mechanics of plates and shells proposed in /7/ and applied later in the theory of laminar shells (/8/, etc.) by the initial equalities (1.3) used, that more adequatly reflect the stress field in the filler of sandwich shells. We also note that within the framework of expressions (1.8) and (1.12), non-linear kinematic relationships of a quadratic approximation can indeed be compiled in place of the linear relationships (1.12). However, their use to construct a linearized theory and to formulate stability problems in order to investigate mixed modes, described in /1/, is not advisable because of the serious complication of their corresponding relationships without considerable refinement.

2. We introduce into our consideration vectors of the given forces and moments

$$\mathbf{\Phi}^{(k)} = \Phi_n^{(k)}\mathbf{n} + \Phi_{n\tau}^{(k)}\boldsymbol{\tau} + \Phi_m^{(k)}\mathbf{m}, \quad \mathbf{L}^{(k)} = L_{n\tau}^{(k)}\mathbf{n} + L_n^{(k)}\boldsymbol{\tau}$$

as applied to the boundary contours $C_{(k)}$ of the middle surfaces of the outer layers, the vectors of the external surface forces and the moments

$$\mathbf{X}_{(k)} = X_{(k)}^i\mathbf{r}_i + X_k^{(3)}\mathbf{m}, \quad \mathbf{M}_{(k)} = M_{(k)}^i\mathbf{r}_i$$

reduced to the surfaces $\sigma_{(k)}$, the bulk force vector in the filler given by the expansion

$$\mathbf{F} = F^i\mathbf{r}_i + F^3\mathbf{m}$$

and also the vector of the surface forces $\mathbf{p}$ given by the expansion

$$\mathbf{p} = p_n\mathbf{n} + p_{n\tau}\boldsymbol{\tau} + p^3\mathbf{m}$$

applied to the filler boundary cut, where $(\mathbf{n}, \boldsymbol{\tau}$ are vectors of the unit normal and tangent to the contour $C \equiv C_{(k)})$.

An expression can be obtained for the work of the above-mentioned external forces on their corresponding possible displacements by using relationships (1.8) and (1.12) ($ds$ is the element of length of the contour line $C$, integration with respect to $z$ is performed everywhere later within the limits $-h$ and $h$, and summation is from $k = 1$ to $k = 2$)

$$\delta A = \iint\limits_{\sigma} \Big[\sum (x_{(k)}^i \delta u_i^{(k)} + x_{(k)}^3 \delta w^{(k)} - m_{(k)}^i \delta\omega_i^{(k)}) + Q^i d_{is}\delta q^s - E_3^{-1} Q^3 \nabla_i \delta q^i\Big] d\sigma + \tag{2.1}$$

$$\int\limits_C \Big[\sum (\varphi_n^{(k)} \delta u_n^{(k)} + \varphi_{n\tau}^{(k)} \delta u_\tau^{(k)} + \varphi_m^{(k)} \delta w^{(k)} + l_{n\tau}^{(k)} \delta\omega_\tau^{(k)} - l_n^{(k)} \delta\omega_n^{(k)}) +$$

$$(P_n n^i + P_{n\tau}\tau^i) d_{is}\delta q^s - E_3^{-1} P_m \nabla_i \delta q^i\Big] ds$$

$$\omega_n^{(k)} = \omega_i^{(k)} n^i, \quad \omega_\tau^{(k)} = \omega_i^{(k)}\tau^i, \quad u_n^{(k)} = u_i^{(k)} n^i, \quad u_\tau^{(k)} = u_i^{(k)}\tau^i$$

$$n^i = \mathbf{n}\mathbf{r}^i, \quad \tau^i = \boldsymbol{\tau}\mathbf{r}^i$$

$$x_{(k)}^3 = X_{(k)}^3 + {}^1/_2 \int F^3 (1 - \delta_{(k)} z/h)\, dz, \quad x_{(k)}^i = X_{(k)}^i + \int F^i f_{(k)}\, dz$$

$$m_{(k)}^i = M_{(k)}^i - \int F^i \chi_{(k)}\, dz, \quad \varphi_n^{(k)} = \Phi_n^{(k)} + \int p_n f_{(k)}\, dz$$

$$\varphi_\tau^{(k)} = \Phi_\tau^{(k)} + \int p_{n\tau} f_{(k)}\, dz, \quad \varphi_m^{(k)} = \Phi_m^{(k)} + {}^1/_2 \int p^3 (1 - \delta_{(k)} z/h)\, dz$$

$$Q^i = \int F^i \chi\, dz, \quad Q^3 = {}^1/_2 \int F^3 (z^2 - h^2)\, dz, \quad l_n^{(k)} = L_n^{(k)} - \int p_n \chi_{(k)}\, dz$$

$$l_{n\tau}^{(k)} = L_{n\tau}^{(k)} + \int p_{n\tau}\chi_{(k)}\, dz, \quad P_n = \int p_n \chi\, dz$$

$$P_{n\tau} = \int p_{n\tau}\chi\, dz, \quad P_m = {}^1/_2 \int p^3 (z^2 - h^2)\, dz$$

If generalized internal forces and moments are introduced into consideration

$$t_{(k)}^{ij} = T_{(k)}^{ij} + \int \sigma^{ij} f_{(k)}\, dz, \quad m_{(k)}^{ij} = M_{(k)}^{ij} - \int \sigma^{ij}\chi_{(k)}\, dz \tag{2.2}$$

$$S_{(k)}^{ij} = {}^1/_2 \int \sigma^{ij} (1 - \delta_{(k)} z/h)\, dz, \quad T^{33} = \int \sigma^{33}\, dz$$

$$M^{33} = \int \sigma^{33} z\, dz, \quad N^{ij} = \int \sigma^{ij}\chi\, dz, \quad H^{ij} = {}^1/_2 \int \sigma^{ij} (z^2 - h^2)\, dz$$

in which $T_{(k)}{}^{ij}$, $M_{(k)}{}^{ij}$ are contravariant components of the internal force and moments tensor in the outer layers, reduced to their middle surfaces and the equality $2\sigma^{i3}\delta\varepsilon_{i3} = d_{ki} q^k \delta q^i$, is taken into account, we arrive at the expression

$$\delta U = \iint\limits_{\sigma} \Big[\int (\sigma^{ij}\delta\varepsilon_{ij} + 2\sigma^{i3}\delta\varepsilon_{i3} + \sigma^{33}\delta\varepsilon_{33})\, dz + \sum (T_{(k)}^{ij}\delta\varepsilon_{ij}^{(k)} + M_{(k)}^{ij}\delta\varkappa_{ij}^{(k)})\Big] d\sigma = \tag{2.3}$$

$$\iint\limits_{\sigma} \Big\{\sum (t_{(k)}^{ij}\nabla_i \delta u_j^{(k)} - m_{(k)}^{ij}\nabla_i \delta\omega_j^{(k)} + T_{(k)}^{ij}\omega_i^{(k)}\delta\omega_j^{(k)} -$$

$$[{}^1/_2\delta_{(k)}T^{33} + b_{ij}(T^{ij}_{(k)} + S^{ij}_{(k)})]\,\delta w^{(k)}\} + c_{ij}q^i\delta q^j +$$
$$N^{ij}d_{js}\nabla_i\delta q^s + E_3^{-1}(b_{ij}H^{ij} - M^{33})\nabla_s\delta q^s\Big\}\,d\sigma$$

when taking account of relationships (1.12), (1.2) and (1.7) to calculate the variations in the strain potential energy of the shell as a whole.

Starting from (1.2), (2.1) and (2.3), a Lagrange variational equation of the following kind can be set up after traditional manipulations:

$$\sum (l^{(k)}_{n\tau} - g^{(k)}_{n\tau})\,\delta w^{(k)}|_C + \int\limits_C \Big\{\sum [(\varphi^{(k)}_n - t^{(k)}_n)\,\delta u^{(k)}_n + (\varphi^{(k)}_{n\tau} - t^{(k)}_{n\tau})\,\delta u^{(k)}_\tau + \tag{2.4}$$
$$(\varphi^{(k)}_m - dl^{(k)}_{n\tau}/ds - S^i_{(k)}n_i + dg^{(k)}_{n\tau}/ds)\,\delta w^{(k)} - (l^{(k)}_n - g^{(k)}_n)\,\delta\omega^{(k)}_n] +$$
$$[d_n(P_n - N_n) + d_{n\tau}(P_{n\tau} - N_{n\tau}) - E_3^{-1}(b_{ij}H^{ij} - M^{33} + Q^3)]\,\delta q_n +$$
$$[d_{n\tau}(P_n - N_n) + d_\tau(P_{n\tau} - N_{n\tau})]\,\delta q_\tau - E_3^{-1}P_m\nabla_i\delta q^i\Big\}\,ds +$$
$$\iint\limits_\sigma \Big[\sum (f^i_{(k)}\delta u^{(k)}_i + f^3_{(k)}\delta w^{(k)}) + \mu_i\delta q^i\Big]\,d\sigma = 0$$

$$d_n = d_{ij}n^in^j,\; d_{n\tau} = d_{ij}n^i\tau^j,\; d_\tau = d_{ij}\tau^i\tau^j \tag{2.5}$$
$$N_n = N^{ij}n_in_j,\; N_{n\tau} = N^{ij}n_i\tau_j,\; t^{(k)}_n = t^{ij}_{(k)}n_in_j$$
$$t^{(k)}_{n\tau} = t^{ij}_{(k)}n_i\tau_j,\; g^{(k)}_n = m^{ij}_{(k)}n_in_j,\; g^{(k)}_{n\tau} = -m^{ij}_{(k)}n_i\tau_j$$
$$S^i_{(k)} = \nabla_j m^{ij}_{(k)} + T^{ij}_{(k)}\omega^{(k)}_j + m^i_{(k)},\; q_n = q^in_i,\; q_\tau = q^i\tau_i$$
$$f_{(k)} = \nabla_j t^{ij}_{(k)} - S^j_{(k)}b_j{}^i + x^i_{(k)},\; f^3_{(k)} = \nabla_i S^i_{(k)} + b_{ij}(T^{ij}_{(k)} + S^{ij}_{(k)}) + {}^1/_2\delta_{(k)}h^{-1}T^{33} + x^3_{(k)}$$
$$\mu_s = \nabla_s[E_3^{-1}(b_{ij}H^{ij} - M^{33} + Q^3)] + \nabla_i(d_{js}N^{ij}) - c_{is}q^i + Q^id_{is}$$

A system of eight equilibrium differential equations

$$f_{(k)}{}^i = 0,\; f_{(k)}{}^3 = 0,\; \mu_s = 0 \tag{2.6}$$

and their corresponding boundary conditions at points of the contour $C$

$$\varphi^{(k)}_n = t^{(k)}_n \text{ for } \delta u^{(k)}_n \neq 0,\; \varphi^{(k)}_{n\tau} = t^{(k)}_{n\tau} \text{ for } \delta u^{(k)}_\tau \neq 0 \tag{2.7}$$
$$\varphi^{(k)}_m - dl^{(k)}_{n\tau}/ds = S^i_{(k)}n_i - dg^{(k)}_{n\tau}/ds \text{ for } \delta w^{(k)} \neq 0$$
$$d_n(P_n - N_n) + d_{n\tau}(P_{n\tau} - N_{n\tau}) - E_3{}^{-1}(b_{ij}H^{ij} - M^{33} + Q^3) =$$
$$0 \text{ for } \delta q_n \neq 0,\; d_{n\tau}(P_n - N_n) + d_\tau(P_{n\tau} - N_{n\tau}) = 0 \text{ for } q_\tau \neq 0$$

follows from the variational Eq.(2.4) constructed.

The static conditions $l_{n\tau}{}^{(k)} - g_{n\tau}{}^{(k)} = 0$ for $\delta w^{(k)} \neq 0$ that have the same meaning as in the classical Kirchhoff-Love theory of shells, are appended to the latter at the free angular point of the contour $C_{(k)}$.

Within the framework of the model constructed, there is no possibility of formulating a static boundary condition related by $\nabla_i\delta q^i$, as follows from the contour integral of (2.4), i.e., the component $E_3{}^{-1}P_m\nabla_i\delta q^i$ is not needed in (2.1) and therefore in (2.4) either.

The elasticity relationships set up by the dependence between the force factors $t^{ij}_{(k)}$, $m^{ij}_{(k)}$, $S^{ij}_{(k)}$, $T^{33}$, $M^{33}$, $N^{ij}$, $H^{ij}$ introduced into the consideration and the eight unknown functions $u^{(k)}_i$, $w^{(k)}$, $q^i$ are comprised on the basis of (2.2) using (1.2), (1.7) and (1.12) and elasticity relationships for the outer layers, not presented here, that are described within the framework of the usual Kirchhoff-Love model taking temperature effects into account.

The linearized equations of the theory of elastic stability of sandwich shells within the framework of the Euler concept of the existence of two adjacent equilibrium modes at the instant of buckling are established by starting from the derived set of relationships by the traditional method. They enable the mixed buckling modes described in /1/ to be investigated in a refined formulation.

3. If we turn to the initial assumptions (1.3), $\sigma^{ij} = 0$, then (2.2) take the form

$$t^{ij}_{(k)} = T^{ij}_{(k)},\; m^{ij}_{(k)} = M^{ij}_{(k)},\; S^{ij}_{(k)} = N^{ij} = H^{ij} = 0$$

Since

$$T^{33} = \int \sigma^{33}dz = E_3(w^{(2)} - w^{(1)}),\quad M^{33} = -{}^2/_3h^3\nabla_iq^i + M^{33}_{(t)}$$
$$M^{33}_{(t)} = E_3\int \alpha_3Tzdz$$

here, the six outer layer equilibrium equations following from (2.5) and (2.6)

$$f^i_{(k)} = \nabla_j T^{ij}_{(k)} - S^j_{(k)} b_j{}^i + x^i_{(k)} = 0,\; f^3_{(k)} = \nabla_i S^i_{(k)} + b_{ij} T^{ij}_{(k)} + {}^1/_2 h^{-1} \delta_{(k)} E_3 (w^{(2)} - w^{(1)}) + x^3_{(k)} = 0$$

and the two equations that hold for the filler

$$\mu_s = {}^2/_3 E_3^{-1} h^3 \nabla_s \nabla_i q^i - E_3^{-1} \nabla_s M^{33}_{(t)} + E_3^{-1} \nabla_s Q^3 - c_{is} q^i + d_{is} Q^i = 0$$

turn out not to be interrelated and lose their physical meaning. Therefore, the set of relations constructed in the previous sections does not allow formal passage to the model of a transversely-soft filler. The set of relationships taking account of temperature effects on the shell that is needed for this case can be constructed using the procedure described in /5/.

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# DEFORMATION OF A VISCOELASTIC CYLINDER FASTENED TO A HOUSING UNDER NON-ISOTHERMAL DYNAMIC LOADING*

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The state of stress and strain is determined for a hollow long mechanically incompressible viscoelastic cylinder fastened to an elastic shell. Unlike other publications /1, 2/, the case of non-isothermal dynamic loading is examined. The cylinder material is considered to be physically non-linear and a physically linear viscoelastic medium whose mechanical properties depend considerably on the temperature. The temperature field is inhomogeneous and non-stationary. A change in the inner surface of the cylinder with time is allowed during the loading. The results of the solution enable safe working conditions for the structure under consideration to be found for definite temperature, mechanical, and geometric data. Some characteristic graphs of the contact stress as a function of time are presented in the case of instantaneous delivery of heat to the inner and outer cylinder surfaces.

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